Goldstone boson counting in linear sigma models with chemical potential

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We analyze the effects of finite chemical potential on spontaneous breaking of internal symmetries within the class of relativistic field theories described by the linear sigma model. Special attention is paid to the emergence of "abnormal" Goldstone bosons with quadratic dispersion relation. We show that their presence is tightly connected to nonzero density of the Noether charges, and formulate a general counting rule. The general results are demonstrated on an SU(3) × U(1) invariant model with an SU(3)-sextet scalar field, which describes one of the color-superconducting phases of QCD.

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I. INTRODUCTION

Spontaneous symmetry breaking plays an important role in many areas of physics and encounters a host of fascinating phenomena. The most distinguishing feature of spontaneous symmetry breaking is the presence of soft modes, long-wavelength fluctuations of the order parameter(s), guaranteed by the Goldstone theorem [1, 2].

For low-energy properties of the spontaneously broken symmetry it is important to know the number of the Goldstone bosons (GBs). While for spontaneously broken internal symmetry (space-time symmetries will not be the subject of this paper, see e.g. Ref. [3]) in a Lorentz-invariant field theory it is always equal to the number of broken symmetry generators, the original Goldstone theorem predicts the existence of *at least one* GB. Indeed, there are several examples in nonrelativistic physics where the number of GBs is smaller than one would naively expect. The most profound one is perhaps the ferromagnet where the rotational SO(3) symmetry is spontaneously broken down to SO(2), but only one GB (the magnon) exists.

The issue of GB counting in nonrelativistic field theories was enlightened by Nielsen and Chadha [4]. They showed that the defect in the number of GBs is related to the low-momentum behavior of their dispersion relations. GBs with energy proportional to an odd power of momentum are classified as type-I, and those with energy proportional to an even power of momentum as type-II. The improved counting rule then states that *the number of GBs of type I plus twice the number of GBs of type II is greater or equal to the number of broken generators.*

It should be noted that the form of the dispersion law of the lightest degrees of freedom has important phenomenological consequences, e.g. for the low-temperature thermodynamics of the system. For instance, the heat capacity of a gas of bosons with $E \propto |\mathbf{p}|$ falls down as T^3 for $T \to 0$, while for bosons with $E \propto \mathbf{p}^2$ it is only $T^{3/2}$. If no massless particles are present, the heat capacity is suppressed by factor $e^{-m/kT}$, where m is the mass of the lightest particle.

The interest in the problem of GB counting has been revived recently, mainly thanks to the progress in understanding the phase diagram of quantum chromodynamics. At finite density Lorentz invariance is explicitly broken and GBs with nonlinear (as a matter of fact, generally quadratic) dispersion relations may appear even in a relativistic field theory as a medium effect [5, 6]. Their presence turns out to be connected to the fact that some of the broken Noether charges develop nonzero density in the ground state, as has been observed in various color-superconducting phases of QCD [7, 8] or in a neutron ferromagnet [9].

Schafer et al. [5] have proved the following theorem: if the commutators of all pairs of broken generators have zero ground-state expectation value, then the number of GBs is equal to the number of broken generators. It is therefore clear that the nonzero charge density itself is not sufficient for a quadratic GB to appear. Indeed, the baryon number density does not make any harm to the usual linear GBs in the color superconductors. The corresponding generator must rather be a part of a non-Abelian symmetry group. Our main goal is to show that the opposite to the theorem of Schafer et al. generally holds: *nonzero density of a commutator of two broken generators implies one GB with quadratic dispersion law.*

The paper is organized as follows. The following section is devoted to preparatory considerations: we explain how the quadratic GB is manifested in the Goldstone commutator and sketch its realization in the linear sigma model. In the next part, an example with an SU(3)-sextet condensation is investigated in detail. The general analysis is performed in the last section.

II. PRELIMINARY CONSIDERATIONS

In this section we shall investigate how the quadratic GBs come about, first at the rather general level of the Goldstone commutator and later more explicitly within the linear sigma model.

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A. Goldstone commutator

Let us briefly recall the proof of the Goldstone theorem. Following Ref. [4], we assume there is a local (possibly composite) field $\Phi(x)$ and a broken Noether charge Q such that $\langle 0|[\Phi(x),Q]|0\rangle \neq 0$. Inserting the complete set of intermediate states into the commutator, one arrives at the representation

$$\langle 0|[\Phi(x),Q]|0\rangle = \sum_{n=1}^{l} \left[e^{-iE_{\mathbf{k}}t}\langle 0|\Phi(0)|n_{\mathbf{k}}\rangle\langle n_{\mathbf{k}}|j^0(0)|0\rangle - e^{iE_{-\mathbf{k}}t}\langle 0|j^0(0)|n_{-\mathbf{k}}\rangle\langle n_{-\mathbf{k}}|\Phi(0)|0\rangle\right] \quad \text{at } \mathbf{k}=0, \quad (1)$$

where the index n counts the GBs.

Now assume that we deal with a non-Abelian symmetry group and some of its charges have nonzero density in the ground state. Take as the GB field $\Phi(x)$ the zero component of the Noether current itself, so that $\langle 0|[j^0_a(x),Q_b]|0\rangle = if_{abc}\langle 0|j^0_c(x)|0\rangle$, where f_{abc} is the set of structure constants of the symmetry group. Should this be nonzero, we infer from Eq. (1) that both $\langle 0|j^0_a(0)|n\rangle$ and $\langle n|j^0_b(0)|0\rangle$ must be nonzero for some Goldstone mode n.

The point of the above heuristic argument is that while in Lorentz-invariant theories there is a one-to-one correspondence between the GBs and the broken currents, here a single GB couples to two Noether currents. This explains (not proves, of course) at a very elementary level how the GB counting rule is to be modified in the presence of nonzero charge density.

One should perhaps note that the Nielsen–Chadha counting rule is formulated in terms of the GB dispersion relations rather then charge densities. The connection between these two was clarified by Leutwyler [10], who showed by the analysis of the Ward identities for the broken symmetry, that nonzero density of a non-Abelian charge induces a term in the low-energy effective Lagrangian with a single time derivative. The leading order effective Lagrangian is thus of the Schrödinger type and the energy of the GB is proportional to momentum squared.

B. Goldstone bosons within the linear sigma model

In order to elaborate more on the properties of the GBs, we restrict ourselves from now on to the framework of the linear sigma model, that is a general scalar field theory with quartic self-interaction.

To see how the Goldstone commutator emerges in this language, recall the SU(2) × U(1) invariant model of Schafer et al. [5] and Miransky and Shovkovy [6]. The Lagrangian for the complex doublet field ϕ of mass M in Minkowski space reads

$$\mathcal{L} = D_\mu\phi^\dagger D^\mu\phi - M^2\phi^\dagger\phi - \lambda(\phi^\dagger\phi)^2.$$

Finite density of the statistical system is represented by the chemical potential μ, which enters the Lagrangian in terms of the covariant derivative [11], $D_\mu\phi = (\partial_\mu - i\delta_{0\mu}\mu)\phi$. Upon expanding the covariant derivatives, the Lagrangian becomes

$$\mathcal{L} = \partial_\mu\phi^\dagger\partial^\mu\phi - 2\mu\,\mathrm{Im}\,\phi^\dagger\partial_0\phi + (\mu^2 - M^2)\phi^\dagger\phi - \lambda(\phi^\dagger\phi)^2. \quad (2)$$

For $\mu > M$ the static potential develops a nontrivial minimum and the scalar field condenses. To find the spectrum of excitations at tree level we reparameterize it as

$$\phi = \frac{1}{\sqrt{2}}e^{i\pi_k\tau_k/v}\begin{pmatrix}0\\ v+\varphi\end{pmatrix}, \quad v^2 = \frac{\mu^2-M^2}{\lambda},$$

and look at the bilinear part of the Lagrangian. The crucial contribution comes from the term in Eq. (2) with one time derivative. Upon expanding the exponentials it yields among others the expression

$$-\frac{1}{2}\mu\,\mathrm{Im}\begin{pmatrix}0 & 1\end{pmatrix}[\pi_k\tau_k, \partial_0\pi_l\tau_l]\begin{pmatrix}0\\1\end{pmatrix} = \mu(\pi_1\partial_0\pi_2 - \pi_2\partial_0\pi_1).$$

As will be made clear in the next subsection, it is this term that is responsible for the quadratic dispersion relation of one of the GBs. Its origin from the nonzero density of a commutator of two generators is now made obvious. This is the main idea to be remembered. The necessary technical details will come in the next two sections.

C. Bilinear Lagrangians and dispersion laws

Bilinear Lagrangians with single-time-derivative terms will frequently occur throughout the whole text. It is therefore worthwhile to fix once for all the corresponding excitation spectrum.

The bilinear Lagrangians we will encounter will have the generic form

$$\mathcal{L}_{\text{bilin}} = \frac{1}{2}(\partial_\mu\pi)^2 + \frac{1}{2}(\partial_\mu H)^2 - \frac{1}{2}f^2(\mu)H^2 - g(\mu)H\partial_0\pi. \quad (3)$$

The notation suggests that H is a massive (Higgs) mode whose mass function $f^2(\mu)$ depends on the chemical potential, while π is the Goldstone mode. The excitation spectrum is found from the poles of the two-point Green functions or, equivalently, by solving the condition

$$\det\begin{pmatrix}E^2-\mathbf{p}^2 & +iEg(\mu)\\ -iEg(\mu) & E^2-\mathbf{p}^2-f^2(\mu)\end{pmatrix} = 0.$$

It turns out there is one massive mode, with dispersion relation

$$E^2 = f^2(\mu) + g^2(\mu) + \mathcal{O}(\mathbf{p}^2), \quad (4)$$

and one massless mode, with dispersion relation

$$E^2 = \frac{f^2(\mu)}{f^2(\mu)+g^2(\mu)}\mathbf{p}^2 + \frac{g^4(\mu)}{\left[f^2(\mu)+g^2(\mu)\right]^3}\mathbf{p}^4 + \mathcal{O}(\mathbf{p}^6). \tag{5}$$

Now if $f^2(\mu) > 0$, the Lagrangian (3) indeed describes a massive particle and a GB, whose energy is linear in momentum in the long-wavelength limit. On the other hand, when $f^2(\mu) = 0$, that is when both π and H would correspond to linear GBs in the absence of the chemical potential, the dispersion relation of the gapless mode reduces to $E = \mathbf{p}^2/|g(\mu)|$. This is the sought quadratic Goldstone.

In conclusion, the term with a single time derivative in general mixes the original fields in the Lagrangian. Mixing of a massive mode with a massless one yields one massive particle and one linear GB, mixing of two massless modes results in a massive particle and a quadratic GB [19].

III. LINEAR SIGMA MODEL FOR SU(3)-SEXTET CONDENSATION

As a nontrivial demonstration of the general idea proposed in the previous section, we shall now analyze in detail a particular model of spontaneous symmetry breaking. Consider a scalar field Φ that transforms as a symmetric rank-two tensor under the group SU(3), $\Phi \to U\Phi U^T$. Such a field describes a one-flavor diquark condensate in one of the superconducting phases of QCD [12].

In addition to the SU(3) group, Φ is subject to U(1) transformations corresponding to quark number, $\Phi \to e^{i\theta}\Phi e^{i\theta} = e^{2i\theta}\Phi$. The most general SU(3)×U(1) invariant Lagrangian has the form

$$\mathcal{L} = \operatorname{tr}(D_\mu\Phi^\dagger D^\mu\Phi) - M^2\operatorname{tr}\Phi^\dagger\Phi - a\operatorname{tr}(\Phi^\dagger\Phi)^2 - b(\operatorname{tr}\Phi^\dagger\Phi)^2. \tag{6}$$

The quark-number U(1) has been assigned chemical potential μ so that $D_0\Phi = (\partial_0 - 2i\mu)\Phi$. The parameters a, b are constrained by the requirement of boundedness of the static potential [12]. It is necessary that either both are non-negative (and at least one of them nonzero), or $a < 0$ and $b > |a|$, or $b < 0$ and $a > 3|b|$.

A. Minimum of the static potential

We start our analysis with a careful inspection of the static potential,

$$V(\Phi) = -(4\mu^2 - M^2)\operatorname{tr}\Phi^\dagger\Phi + a\operatorname{tr}(\Phi^\dagger\Phi)^2 + b(\operatorname{tr}\Phi^\dagger\Phi)^2. \tag{7}$$

A potential of the same type has been analyzed by Iida and Baym [13]. In their case, however, the global symmetry was different, and we therefore provide full details.

When $4\mu^2 - M^2 > 0$, the stationary point $\Phi = 0$ becomes unstable and a new, nontrivial minimum appears [20]. The stationary-point condition reads

$$\Phi\left(-4\mu^2 + M^2 + 2a\Phi^\dagger\Phi + 2b\operatorname{tr}\Phi^\dagger\Phi\right) = 0. \tag{8}$$

Before going into detailed solution of this equation we note that by multiplying Eq. (8) from left by $\Phi^\dagger$ and taking the trace, the stationary-point value of the potential (7) is found to be

$$V_{\text{stat}} = -\frac{1}{2}(4\mu^2 - M^2)\operatorname{tr}\Phi^\dagger\Phi.$$

Any nontrivial stationary point of the potential is thus energetically more favorable than the perturbative vacuum $\Phi = 0$. We are, however, obliged to find a stable ground state, that is the absolute minimum of the potential.

We now make use of the fact that the field Φ can always be brought by a suitable SU(3) × U(1) transformation to the standard form, which is a real diagonal matrix with non-negative entries [14]. Eq. (8) then splits into three conditions and it is easy to see that all nonzero diagonal elements acquire the same value, denoted here by Δ.

Let there be n of them, $n = 1, 2, 3$. Eq. (8) implies

$$\Delta^2 = \frac{1}{2}\frac{4\mu^2 - M^2}{a + bn}, \quad V_{\text{stat}} = -\frac{1}{4}\frac{(4\mu^2 - M^2)^2}{b + \frac{a}{n}}.$$

To find the absolute minimum of the potential, it remains to minimize this expression with respect to n.

For $a > 0$ the minimum occurs at $n = 3$, and Φ is proportional to the unit matrix, $\Phi = \Delta\mathbb{1}$, where

$$\Delta^2 = \frac{1}{2}\frac{4\mu^2 - M^2}{a + 3b}.$$

The SU(3) × U(1) symmetry is broken down to SO(3).

For $a < 0$ the potential is minimized by $n = 1$, that is Φ is diagonal with a single nonzero entry and is conventionally chosen to be $\Phi = \operatorname{diag}(0, 0, \Delta)$, where now

$$\Delta^2 = \frac{1}{2}\frac{4\mu^2 - M^2}{a + b}.$$

The unbroken subgroup is now SU(2) × U(1).

For $a = 0$ the local minima corresponding to different n are degenerate since in that case, the Lagrangian (6) is invariant under an enhanced SU(6) × U(1) symmetry, treating Φ as a fundamental sextet. Nonzero ground-state expectation value of Φ breaks this symmetry to SU(5) × U(1). As we shall see, such an enhanced symmetry leads to an increased number of GBs with quadratic dispersion relation [15].

B. Noether currents and charge densities

Having found the vacuum configuration of the scalar field, we are ready to reparameterize it and find the excitation spectrum from the bilinear part of the Lagrangian.

Before doing that, we evaluate the ground-state densities of the Noether charges in order to make *a priori* predictions about the nature of the GBs.

The infinitesimal SU(3) × U(1) transformation of Φ has the generic form $\delta\Phi = i\theta_k(\lambda_k\Phi + \Phi\lambda_k^T)$, where the λ_k stands for the Gell-Mann matrices ($k = 1, \ldots, 8$) and the unit matrix ($k = 0$), respectively. The corresponding Noether currents are

$$j_k^\mu = -i\,\mathrm{tr}\left[D^\mu\Phi^\dagger\left(\lambda_k\Phi + \Phi\lambda_k^T\right) - \mathrm{h.c.}\right].$$

Taking a generic static field configuration to be $\Phi = \mathrm{diag}(\Delta_1, \Delta_2, \Delta_3)$ results in the charge densities

$$\begin{aligned} j_0^0 &= 8\mu(\Delta_1^2 + \Delta_2^2 + \Delta_3^2), \\ j_3^0 &= 8\mu(\Delta_1^2 - \Delta_2^2), \\ j_8^0 &= \frac{8}{\sqrt{3}}\mu(\Delta_1^2 + \Delta_2^2 - 2\Delta_3^2). \end{aligned}$$

In the SO(3) symmetric phase ($a > 0$), all generators but the U(1) quark number have zero density. As this is an Abelian generator, we expect six linear GBs corresponding to the six broken generators $\mathbb{1}, \lambda_1, \lambda_3, \lambda_4, \lambda_6, \lambda_8$. In the $a < 0$ case, the densities of λ_0 and λ_8 are nonzero. This means that the commutators $[\lambda_4, \lambda_5]$ and $[\lambda_6, \lambda_7]$ have nonzero ground-state density. With regard to the general discussion above, we thus expect two quadratic GBs corresponding to pairs (λ_4, λ_5) and (λ_6, λ_7), and one linear GB of the generator λ_8.

C. The $a > 0$ case

We shall now proceed to the calculation of the mass spectrum of the $a > 0$ phase. We could do well with just shifting Φ by its vacuum expectation value $\Delta\mathbb{1}$, but this would complicate the identification of the massless modes. It is more convenient, and physical, to find such a parameterization that the GBs disappear from the static potential.

To that end, recall that the field $\Phi(x)$ (now coordinate-dependent) can be brought to the diagonal form by a suitable SU(3) × U(1) transformation. In other words, it may be written as

$$\Phi(x) = e^{2i\theta(x)}U(x)D(x)U^T(x),$$

where $U(x) \in \mathrm{SU}(3)$ and $D(x)$ is real, diagonal and non-negative. Now the unitary matrix U can be (at least in the vicinity of unity) expressed as a product $U = VO$, $O \in \mathrm{SO}(3)$ being an element of the unbroken subgroup and V being built from the broken generators, $V = e^{i\pi_k\lambda_k}$, $k = 1, 3, 4, 6, 8$. A simple observation that $O(x)D(x)O^T(x)$ is the general parameterization of a real symmetric matrix leads to the final prescription,

$$\Phi(x) = e^{2i\theta(x)}V(x)\left[\Delta\mathbb{1} + \varphi(x)\right]V^T(x).$$

The real symmetric matrix φ contains six massive modes, while V contains five GBs. With θ this is altogether twelve degrees of freedom, as it should for Φ is a complex symmetric 3×3 matrix.

It is now straightforward, though somewhat tedious, to plug this parameterization into the Lagrangian (6) and expand to the second order in the fields. Omitting details of the calculations, we just report on the results.

The full static potential (up to a constant term – the vacuum energy density) becomes

$$\begin{aligned} V(\Phi) &= 4\Delta^2\left[a\,\mathrm{tr}\,\varphi^2 + b(\mathrm{tr}\,\varphi)^2\right] \\ &\quad + 4\Delta\left(a\,\mathrm{tr}\,\varphi^3 + b\,\mathrm{tr}\,\varphi\,\mathrm{tr}\,\varphi^2\right) + a\,\mathrm{tr}\,\varphi^4 + b(\mathrm{tr}\,\varphi^2)^2. \end{aligned}$$

The bilinear Lagrangian turns out to be (we use the notation $V = e^{i\Pi}$)

$$\begin{aligned} \mathcal{L}_{\mathrm{bilin}} &= 12\Delta^2(\partial_\mu\theta)^2 + 4\Delta^2\,\mathrm{tr}(\partial_\mu\Pi)^2 + \mathrm{tr}(\partial_\mu\varphi)^2 \\ &-4\Delta^2\left[a\,\mathrm{tr}\,\varphi^2 + b(\mathrm{tr}\,\varphi)^2\right] - 16\mu\Delta\left[\partial_0\theta\,\mathrm{tr}\,\varphi + \mathrm{tr}(\varphi\partial_0\Pi)\right]. \end{aligned}$$

The kinetic terms are brought to the canonical form by a simple rescaling of the fields, upon which the spectrum is readily determined from Eqs. (4) and (5).

The excitations fall into irreducible multiplets of the unbroken SO(3) group. There are two singlets, stemming from the mixing of θ and $\mathrm{tr}\,\varphi$,

$$\begin{aligned} \text{massive mode} \qquad & E^2 = 24\mu^2 - 2M^2 + \mathcal{O}(\mathbf{p}^2), \\ \text{linear GB} \qquad & E^2 = \frac{4\mu^2 - M^2}{12\mu^2 - M^2}\mathbf{p}^2 + \mathcal{O}(\mathbf{p}^4), \end{aligned}$$

and two 5-plets, the mixtures of $(\pi_1, \pi_3, \pi_4, \pi_6, \pi_8)$ and the traceless part of φ,

$$\begin{aligned} \text{massive modes} \quad & E^2 = \frac{(24\mu^2 - 2M^2)a + 48\mu^2 b}{a + 3b} + \mathcal{O}(\mathbf{p}^2), \\ \text{linear GBs} \quad & E^2 = \frac{(4\mu^2 - M^2)a}{(12\mu^2 - M^2)a + 24\mu^2 b}\mathbf{p}^2 + \mathcal{O}(\mathbf{p}^4). \end{aligned}$$

It is easily seen from these formulas that the masses of the massive singlet and the massive 5-plet are connected by

$$m_{\mathbf{1}}^2 = m_{\mathbf{5}}^2 + (4\mu^2 - M^2)\frac{6b}{a + 3b} = m_{\mathbf{5}}^2 + 12\Delta^2 b.$$

The singlet is heavier than the 5-plet for $b > 0$ and vice versa.

The excitation spectrum is plotted in Fig. 1 for the case $M^2 > 0$. Below the phase transition to the Bose–Einstein-condensed phase, the medium-modified dispersion relations are simply $E = \sqrt{\mathbf{p}^2 + M^2} \pm 2\mu$. Right at the transition point, there are six modes with mass $2M$ and six massless ones with dispersion $E = \mathbf{p}^2/4\mu$. As the phase transition is second order, the dispersion relations of all excitation branches must be continuous functions of μ, that is *all GBs become quadratic at the transition point*. This is also easily checked on the broken-symmetry side of the transition. As $2\mu \to M+$, the phase velocities of the linear GBs tend to zero, and their dispersions become quadratic.

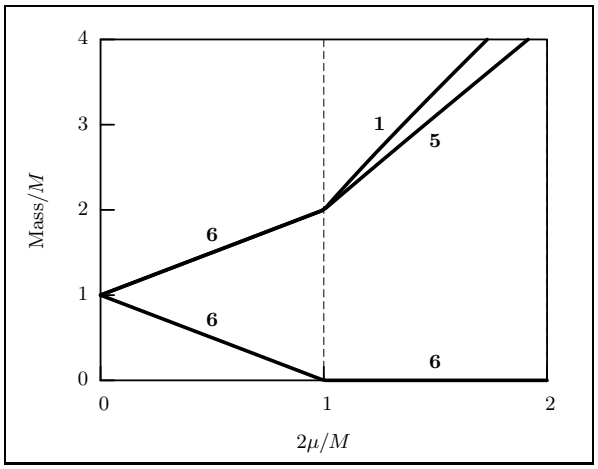

FIG. 1: Mass spectrum as a function of the chemical potential for $a > 0$. The boldface-typed numbers denote the degeneracies of the excitation branches. To obtain numerical results, particular values $a = b = 1$ were chosen.

Note that also for $a = 0$ the dispersion relation of the GB 5-plet becomes quadratic, $E = \mathbf{p}^2/4\mu$. This is in accord with the enhanced SU(6) × U(1) symmetry of the Lagrangian. There are altogether eleven broken generators of the coset SU(6)/SU(5), one linear GB and five quadratic ones [forming now the 5-plet of the unbroken SU(5)], and the Nielsen–Chadha counting rule is thus satisfied.

D. The $a < 0$ case

We use the same method for parameterization of Φ as in the previous case. This time we write $\Phi(x) = U(x)D(x)U^T(x)$, where $U(x) \in \mathrm{SU}(3) \times \mathrm{U}(1)$. Next perform the decomposition $U = e^{i\Pi}U'$, where $\Pi = \pi_k\lambda_k$, $k = 4, 5, 6, 7, 8$, and U' belongs to the unbroken subgroup SU(2) × U(1). Since $U'(x)D(x)U'^T(x)$ is block-diagonal with a complex symmetric 2×2 matrix in the upper-left corner, we arrive at the parameterization

$$\Phi(x) = e^{i\Pi(x)}\left[\mathrm{diag}(0,0,\Delta) + \Sigma(x)\right]e^{i\Pi^T(x)}, \quad \Sigma(x) = \left(\begin{array}{c|c} \sigma(x) & \\ \hline & H(x) \end{array}\right).$$

Here H is a real field and σ is a complex symmetric 2×2 matrix. These two embody the massive modes that survive in the static potential,

$$V(\Phi) = 4\Delta^2(a+b)H^2 - 2\Delta^2 a\,\mathrm{tr}\,\sigma^\dagger\sigma + 4\Delta(a+b)H^3 + (a+b)H^4 + 4\Delta bH\,\mathrm{tr}\,\sigma^\dagger\sigma + 2bH^2\,\mathrm{tr}\,\sigma^\dagger\sigma + a\,\mathrm{tr}(\sigma^\dagger\sigma)^2 + b(\mathrm{tr}\,\sigma^\dagger\sigma)^2.$$

The bilinear part of the Lagrangian reads

$$\mathcal{L}_{\mathrm{bilin}} = \mathrm{tr}(\partial_\mu\sigma^\dagger\partial^\mu\sigma) + (\partial_\mu H)^2 + 2\Delta^2(\partial_\mu\Pi\partial^\mu\Pi)_{33} + 2\Delta^2(\partial_\mu\Pi_{33})^2 - 4\Delta^2(a+b)H^2 + 2\Delta^2 a\,\mathrm{tr}\,\sigma^\dagger\sigma$$
$$- 16\mu\Delta H\partial_0\Pi_{33} - 4\mu\Delta^2\,\mathrm{Im}[\Pi, \partial_0\Pi]_{33} - 4\mu\,\mathrm{Im}\,\mathrm{tr}\,\sigma^\dagger\partial_0\sigma.$$

The excitations are again organized in multiplets of the unbroken SU(2) × U(1). H and π_8 mix to form two singlets,

$$\text{massive mode} \qquad E^2 = 24\mu^2 - 2M^2 + \mathcal{O}(\mathbf{p}^2),$$
$$\text{linear GB} \qquad E^2 = \frac{4\mu^2 - M^2}{12\mu^2 - M^2}\mathbf{p}^2 + \mathcal{O}(\mathbf{p}^4),$$

and the pairs (π_4, π_5) and (π_6, π_7) give rise to a doublet of massive modes and a doublet of massless ones,

$$\text{massive modes} \qquad E^2 = 16\mu^2 + \mathcal{O}(\mathbf{p}^2),$$
$$\text{quadratic GBs} \qquad E^2 = \frac{\mathbf{p}^4}{16\mu^2} + \mathcal{O}(\mathbf{p}^6).$$

The matrix σ represents two triplets of massive particles. The part of the bilinear Lagrangian containing σ may be rewritten as

$$\mathcal{L}_\sigma = \mathrm{tr}(D_\mu\sigma^\dagger D^\mu\sigma) - (4\mu^2 + 2\Delta^2|a|)\,\mathrm{tr}\,\sigma^\dagger\sigma,$$

which immediately implies the dispersion relations

$$E = \sqrt{4\mu^2 + 2\Delta^2|a|} \pm 2\mu + \mathcal{O}(\mathbf{p}^2).$$

The mass spectrum is shown in Fig. 2. The unbroken-phase part of the spectrum is the same as in the $a > 0$ case, since for $2\mu < M$ the *tree-level* masses of the particles do not depend at all on the quartic potential, i.e. the parameters a, b. Also, the same remark about the continuity of the dispersion relations across the phase transition applies.

Again, in the limit $a = 0$, the lighter of the two triplets in σ becomes a triplet of quadratic GBs, and joins the other two quadratic GBs to form the full SU(5) 5-plet.

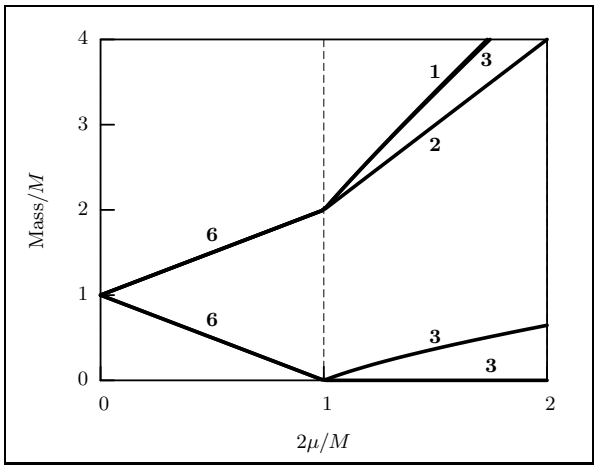

FIG. 2: Mass spectrum as a function of the chemical potential for $a < 0$. The singlet and triplet lines are so close that they almost coincide, but they are not degenerate. The spectrum is plotted for $a = -0.5$ and $b = 1$.

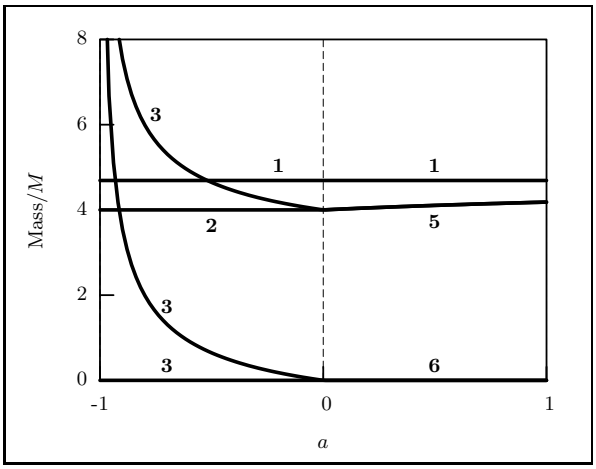

FIG. 4: Mass spectrum as a function of a. The graph is plotted for $\mu = M$ and $b = 1$. The potential is unstable for $a < -1$. The singular behavior of the masses of σ is due to divergence of Δ towards the stability limit of the potential.

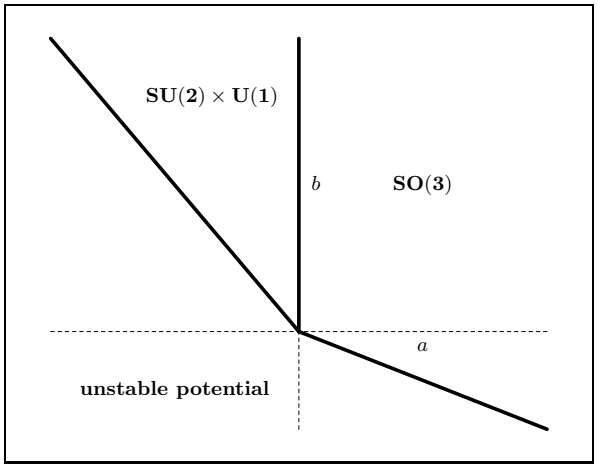

FIG. 3: Phase diagram of the linear sigma model for SU(3)-sextet condensation. The phases are labeled by the symmetry of the ground state. The line of second order phase transition at $a = 0, b > 0$ has SU(5) × U(1) symmetry.

To summarize our results, the theory described by the Lagrangian (6) has two different ordered phases, both occurring at $4\mu^2 > M^2$, distinguished by the symmetry of the ground state. The corresponding phase diagram in the (a, b) plane is displayed in Fig. 3.

As the excitations above the ordered ground state are grouped into irreducible multiplets of the unbroken symmetry, it is interesting to find out how the structure of these multiplets changes across the phase transition from one ordered phase to the other. In Fig. 4 we show the dependence of the masses on the parameter a at constant chemical potential. The masses are continuous functions of a as the transition is second order.

As a final remark we note that in the original application of Ref. [12], the field Φ represented a diquark condensate and the SU(3) was the color gauge group of QCD. One might wonder whether the usual Higgs mechanism for gauge boson masses survives when there are fewer GBs than the number of broken generators, because of the presence of quadratic GBs. This question was answered affirmatively by Gusynin et al. [16], and there is therefore no need to worry about the fate of gluons.

IV. GENERAL ANALYSIS

In this section we shall collect experience gained by solving particular examples and set out for a general analysis. We will find out, with some effort, that the ideas sketched in Sec. II and demonstrated in Sec. III have a straightforward generalization to a whole class of theories. It is understood, however, that we shall all the time stay in the framework of the linear sigma model, and at the tree level. The possibilities of further progress are discussed in the conclusions.

A. Chemical potential and global symmetry

As the starting point we shall address the question what is the most general symmetry of a theory with nonzero chemical potential.

Let the microscopic theory possess a global continuous symmetry with the corresponding conserved Noether charges. The physical meaning of the chemical potential μ is that we wish to fix the statistical average of a conserved charge, say Q. This is technically achieved by

introducing the grandcanonical ensemble and replacing the microscopic Hamiltonian H with $H - \mu Q$.

It is now clear that by adding the chemical potential, we break explicitly all Noether charges that do not commute with Q. This is the technical realization of the physically intuitive fact that we cannot keep simultaneously fixed the values of two noncommuting operators (i.e. incompatible observables).

This simple observation implies that, as far as *exact* symmetry is concerned, chemical potential is always assigned to a generator that commutes with all others, that is to a U(1) factor of the exact global symmetry group.

Of course, when the symmetry of the microscopic theory is non-Abelian, then adding of the chemical potential generally produces a number of approximately conserved charges (at least for small μ) that generate approximate symmetries. These may also be spontaneously broken, resulting in the corresponding set of pseudo-Goldstone bosons. Throughout this paper we are, however, concerned only with true GBs, and therefore only the exact global symmetry will be considered.

It is also interesting to find out how the Abelian nature of the charge equipped with chemical potential is manifested in the Lagrangian formalism. There, as already mentioned, chemical potential enters the Lagrangian in terms of the covariant derivative of "matter" fields [11].

The Lagrangian can be made formally *gauge-invariant* by introducing an external gauge field A_μ. Provided the matter fields ϕ transform under the symmetry group linearly as $\phi \to U\phi$, A_μ transforms as usual as $A_\mu \to UA_\mu U^{-1} + iU\partial_\mu U^{-1}$. Now the exact symmetry is such that the Lagrangian is invariant under the *global* transformation of the matter fields with A_μ *fixed* at $A_\mu = (\mu Q, 0, 0, 0)$. This is possible only when $A_\mu = UA_\mu U^{-1}$. We thus again arrive at the conclusion that the generator being assigned chemical potential must commute with all others.

B. Linear sigma model

Now consider a general linear sigma model defined by the Lagrangian

$$\mathcal{L} = D_\mu \phi^\dagger D^\mu \phi - V(\phi). \tag{9}$$

Here ϕ denotes a set of complex [21] scalar fields that form a (possibly reducible) multiplet of the exact global symmetry group G, i.e. span the target space of a (possibly reducible) representation of G, say $\mathcal{R}$. $V(\phi)$ is the most general G-invariant static potential containing terms up to the fourth power of ϕ, and the covariant derivative is given by $D_\mu\phi = (\partial_\mu - iA_\mu)\phi$. A_μ is the constant external field that incorporates chemical potential for one or more U(1) factors of G, and is eventually set to $A_\mu = (\sum_i \mu_i Q_{i\mathcal{R}}, 0, 0, 0)$, where the Q_i's are the U(1) generators, the subscript $\mathcal{R}$ denoting the image in the representation $\mathcal{R}$.

Upon expanding the covariant derivatives Eq. (9) takes the form

$$\mathcal{L} = \partial_\mu \phi^\dagger \partial^\mu \phi - 2\,\mathrm{Im}\,\phi^\dagger A^\mu \partial_\mu \phi - V_{\mathrm{eff}}(\phi), \tag{10}$$

the effective μ-dependent potential being $V_{\mathrm{eff}}(\phi) = V(\phi) - \phi^\dagger A^\mu A_\mu \phi$.

Spontaneous symmetry breaking occurs when $V_{\mathrm{eff}}(\phi)$ develops a nontrivial minimum at some $\phi = \phi_0$. In order to elucidate the physical content of such a theory, it is necessary to conveniently parameterize the field ϕ.

We stress the generality of the parameterization method suggested and applied in Sec. III. One first writes $\phi(x) = U_{\mathcal{R}}(x)\phi_{\mathrm{std}}(x)$, where ϕ_{std} is a standard form to which the field ϕ can always be brought by a suitable transformation $U \in \mathrm{G}$. Next $U_{\mathcal{R}}$ is factorized as $U_{\mathcal{R}} = e^{i\Pi} U'_{\mathcal{R}}$, where Π is a linear combination of the broken generators (or more precisely, their $\mathcal{R}$-images) and U' belongs to the unbroken subgroup H. The final step is to identify $U'_{\mathcal{R}}(x)\phi_{\mathrm{std}}(x)$ with a certain representation of H and parameterize it linearly as $\phi_0 + H(x)$. $H(x)$ is going to be the multiplet of massive (Higgs) fields. We therefore invoke the parameterization

$$\phi(x) = e^{i\Pi(x)}\left[\phi_0 + H(x)\right]. \tag{11}$$

In order to specify the transformation properties of H, recall that the GBs transform linearly in the adjoint representation of the unbroken subgroup [17], i.e. $\Pi \to U'_{\mathcal{R}} \Pi U'^{-1}_{\mathcal{R}}$ for any $U' \in \mathrm{H}$. As a consequence, $H = e^{-i\Pi}\phi - \phi_0$ transforms as $H \to U'_{\mathcal{R}} H$, since ϕ_0 is an H-singlet.

To summarize, H transforms in the representation $\mathcal{R}$ truncated to the subgroup H, and the multiplets of the massive modes are therefore found in the decomposition of $\mathcal{R}$ into irreducible representations of H.

For instance, in our case $a > 0$ the symmetric rank-two tensor representation of SU(3) splits under the SO(3) subgroup into a traceless symmetric rank-two tensor and a singlet. On the other hand, in the $a < 0$ case it yields a symmetric rank-two tensor of SU(2) (the field σ) plus a singlet.

As an aside let us remark that the physical spectrum of the theory of course does not depend on the parameterization chosen for the field ϕ. What if we chose e.g. the linear parameterization mentioned (and abandoned) above in Sec III C? Instead of Eq. (11), we would then have analogously

$$\phi(x) = \phi_0 + H(x) + i\Pi(x)\phi_0. \tag{12}$$

It is easy to see that the *bilinear* terms in the Lagrangian with one or two derivatives come out identical as for the parameterization (11). The reason is that the only difference stemming from the nonlinear structure of $e^{i\Pi}$ could possibly come in the form $\phi_0^\dagger A^\mu \partial_\mu \Pi^2 \phi_0$, but this is real and therefore it drops out of the Lagrangian (10).

The only difficulty with the linear parameterization (12) is that the GBs do not disappear automatically from

the static potential. Instead, we have to use explicitly the G-invariance to show that Π disappears from the *bilinear* (mass) part of the potential.

Upon the field redefinition as in Eq. (11), the effective potential $V_{\rm eff}$ becomes (up to a constant term)

$$V_{\rm eff}(\phi) = V(\phi_0 + H) - (H^\dagger A^\mu A_\mu H + 2\,\mathrm{Re}\, H^\dagger A^\mu A_\mu \phi_0).$$

As we are expanding the potential about its absolute minimum, the additional term linear in H is right enough to cancel a similar term coming from $V(\phi_0 + H)$. We are interested in the bilinear part of the potential, $V_{\rm bilin}(H)$, which determines the mass term for H.

Now we analyze the first two terms of the Lagrangian (10). The two-derivative term yields the bilinear contribution

$$\partial_\mu H^\dagger \partial^\mu H + \phi_0^\dagger \partial_\mu \Pi \partial^\mu \Pi \phi_0 + 2\,\mathrm{Im}\,\phi_0^\dagger \partial_\mu \Pi \partial^\mu H. \qquad (13)$$

The first two terms in Eq. (13) are the expected kinetic terms for the Higgs and Goldstone fields, respectively. The GB term, however, asks for a check that it is nondegenerate.

Let $\Pi(x) = \pi_k(x) T_k$, T_k being the set of broken generators. The GB kinetic term becomes $\partial_\mu \pi_k \partial^\mu \pi_l \phi_0^\dagger T_k T_l \phi_0 = \frac{1}{2} \partial_\mu \pi_k \partial^\mu \pi_l \phi_0^\dagger \{T_k, T_l\} \phi_0$. The matrix $\phi_0^\dagger \{T_k, T_l\} \phi_0$ is real and symmetric and may be chosen, by taking an appropriate basis of broken generators, diagonal. It is obviously nondegenerate, as necessary in order to have kinetic terms for all the GBs, since otherwise $\phi_0^\dagger T_k T_k \phi_0 = 0$ for some T_k, implying that T_k is in fact not broken.

The third term in Eq. (13) eventually turns out to be zero. Nevertheless, as other terms of a similar structure will be dealt with in the following, we shall analyze it in detail. The crucial point is the way various fields transform under the unbroken subgroup H. Virtually all information about the structure of the bilinear Lagrangian may be obtained by a proper decomposition of the representation $\mathcal{R}$ into irreducible representations of H, and making repeated use of the Wigner–Eckart theorem.

Now when H and Π belong to different representations of H, the Wigner–Eckart theorem immediately tells us that the last term of Eq. (13) vanishes. There is, however, a subtle exception to this argument. As $\mathcal{R}$ is a complex representation, real representations of H are doubled in its decomposition. The reason is that when the set of vectors χ_k constitute the basis of a real representation of H, the vectors $i\chi_k$ form an independent basis of an equivalent representation.

It may be that H and Π (or $\Pi\phi_0$) are such doubles. This happens, for instance, for the two 5-plets in Sec. III C. In such a case, however, $\phi_0^\dagger \partial_\mu \Pi \partial^\mu H$ is real and, again, does not contribute to Eq. (13).

The single-derivative term in Eq. (10) gives, after a short manipulation, the bilinear terms

$$-2\,\mathrm{Im}\, H^\dagger A^\mu \partial_\mu H - 4\,\mathrm{Re}\, H^\dagger A^\mu \partial_\mu \Pi \phi_0 - \mathrm{Im}\,\phi_0^\dagger A^\mu [\Pi, \partial_\mu \Pi] \phi_0. \qquad (14)$$

Throughout the calculation we made use of the fact that A^μ is a U(1) generator, and therefore commutes with Π.

Putting together all the pieces of Eqs. (13) and (14), we arrive at our main result – the bilinear Lagrangian for a general linear sigma model,

$$\mathcal{L}_{\rm bilin} = \partial_\mu H^\dagger \partial^\mu H - V_{\rm bilin}(H) - 2\,\mathrm{Im}\, H^\dagger A^\mu \partial_\mu H + \phi_0^\dagger \partial_\mu \Pi \partial^\mu \Pi \phi_0 - 4\,\mathrm{Re}\, H^\dagger A^\mu \partial_\mu \Pi \phi_0 - \mathrm{Im}\,\phi_0^\dagger A^\mu [\Pi, \partial_\mu \Pi] \phi_0. \qquad (15)$$

This formula contains all information about the particle spectrum of the theory, and the rest of the section is therefore devoted to its analysis.

C. Discussion of the results

There are altogether three terms with a single time derivative in Eq. (15). The term $\mathrm{Im}\, H^\dagger A^\mu \partial_\mu H$ causes splitting of the masses of the massive modes. The term $\mathrm{Re}\, H^\dagger A^\mu \partial_\mu \Pi \phi_0$ mixes massive and massless modes and, according to Sec. II C produces linear GBs. Finally, the term $\mathrm{Im}\,\phi_0^\dagger A^\mu [\Pi, \partial_\mu \Pi] \phi_0$ mixes the Goldstone fields and gives rise to the quadratic Goldstones.

With the Wigner–Eckart theorem at hand it is easy to check that each of the elementary fields appears in at most one of the three single-derivative terms. This fact essentially reduces the analysis of the Lagrangian (15) to the model two-field problem discussed in Sec. II C.

To prove it note that the mixing term $\mathrm{Re}\, H^\dagger A^\mu \partial_\mu \Pi \phi_0$ can be nonzero only when H and Π are the two copies of the doubled real representation of H. Now the real multiplet H gives real $H^\dagger A^\mu \partial_\mu H$, and therefore does not contribute to the mixing of the massive modes. The real multiplet Π analogously does not contribute to $\mathrm{Im}\,\phi_0^\dagger A^\mu [\Pi, \partial_\mu \Pi] \phi_0$ as a consequence of the analysis that follows.

As the main concern of this paper is Goldstone boson counting, we shall now concentrate on the last term of Eq. (15), which produces the quadratic GBs.

First, it is clear that our suspicion about the connection between the quadratic GBs and nonzero charge densities was right. For by the very same method as in Sec. III B we derive the Noether current corresponding to the conserved charge T,

$$j_T^\mu = -i(D^\mu \phi^\dagger T \phi - \text{h.c.}),$$

and the ground-state density of T is

$$j_T^0 = 2\phi_0^\dagger A^0 T \phi_0. \qquad (16)$$

The last term of Eq. (15) is therefore indeed proportional to the ground-state density of the commutator of two generators.

We may now in the general case proceed as in Sec. III, that is find the ground state, calculate the Noether charge densities, and make a definite prediction for the

particle spectrum. We can, however, do even better, at least a bit.

We need not calculate the charge densities explicitly to say, which of the generators may produce quadratic GBs. It is obvious from Eq. (16) that only such a generator T may acquire nonzero density, which is a singlet of the unbroken subgroup H. We therefore just have to decompose the adjoint representation of G into irreducible representations of H and look for spontaneously broken singlets.

As in the examples above, we next choose such a basis that all the generators with nonzero density mutually commute. This ensures that they can be completed to form the Cartan subalgebra of the Lie algebra of G. Following the standard root decomposition of Lie algebras (see e.g. Ref. [18]), the rest of generators group into pairs whose commutator lies in the Cartan subalgebra. They are the lowering and raising operators or their hermitian linear combinations, and together with their commutator span an SU(2) subalgebra of G.

The point of this procedure is that only pairs of Goldstone fields are then mixed by the single-derivative term $\mathrm{Im}\,\phi_0^\dagger A^\mu[\Pi, \partial_\mu \Pi]\phi_0$ and the excitation spectrum may be fully described with the help of the simple two-field bilinear Lagrangian (3). Consequently, the quadratic GBs count as one per each pair of generators whose commutator develops nonzero ground-state density.

The feasibility of such a pairing also follows from group theory and the Wigner–Eckart theorem. As the commutator of the two generators is to be an H-singlet, they must come from the same irreducible representation of H.

To briefly conclude this section, we once again emphasize the fact that almost all we need to know about the excitation spectrum of the general linear sigma model (9) may be extracted from the bilinear Lagrangian (15) by simple group theory. We decompose the adjoint representation of G with respect to the unbroken subgroup H to determine the multiplet structure of the Goldstones. The remaining H-multiplets in the decomposition of the representation $\mathcal{R}$ of the scalar field ϕ are the massive modes.

The quadratic GBs are discovered with the knowledge of the ground-state densities of the broken generators. Without further calculation, we can even determine their dispersion relations. Making use of the continuity of the dispersion relations across the phase transition and the known dispersion relations in the unbroken phase, we may assert that the quadratic GB dispersion relation is generically of the form $E = \mathbf{p}^2/2\mu Q$, where Q is the charge of the GB field under the U(1) subgroup equipped with the chemical potential.

V. CONCLUSIONS

We have analyzed spontaneous breaking of internal symmetries in the framework of the relativistic linear sigma model with finite chemical potential. Our prime motivation was to establish a counting rule for Goldstone bosons in view of the fact that explicit breaking of Lorentz invariance by medium effects may cause the number of GBs to differ from the number of broken symmetry generators.

Our results confirm the Nielsen–Chadha counting rule. We show that the GBs have either linear or quadratic dispersion law at low momentum, and that the number of the first plus twice the number of the second gives exactly the number of broken generators.

In addition, we find a criterion which gives in a purely algebraic way the number of quadratic GBs, the only necessary input being the structure of the ground state. *There is one quadratic GB for each pair of generators, whose commutator has nonzero ground-state density.*

However, despite the generality of our results, many open questions still remain. First, we stress the fact that we work all the time at the tree level. It would be interesting to know the effect of radiative corrections on the details of the spectrum. On the other hand, it seems that at least the dispersion relations of the quadratic GBs are rather generic as they depend only on the chemical potential in a very simple way. There might be a more robust, nonperturbative method to determine them, which relies only on the broken symmetry, and does not depend on the details of the dynamics of symmetry breaking.

Second, we worked within the linear sigma model as it is easy to manipulate perturbatively once the scalar field has been properly shifted to its new ground state. It may happen that our results are valid generally for relativistic theories with chemical potential. At least the argument presented in Sec. II A that clarifies the connection between the charge densities and the GB counting, suggests such a possibility.

As adding chemical potential breaks Lorentz invariance in a very particular way, it might be possible to strengthen the Nielsen–Chadha counting rule at the cost of limiting its validity to a smaller class of theories. Even such a theorem would, however, find many applications on relativistic many-particle systems. We hope that our future work will help to find the answer to these questions.

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[19] There is an exception to the first case: right at the phase transition point, $f^2(\mu) = 0$ even for the Higgs mode, the phase velocity of the linear GB goes to zero, and its dispersion relation becomes quadratic [5].

[20] It should be stressed that our analysis applies to both signs of M^2. For $M^2 > 0$ the model describes relativistic Bose–Einstein condensation, which occurs at $2\mu > M$. For $M^2 < 0$ the Lagrangian (6) represents simply a spontaneously broken symmetry at finite density.

[21] Real representations are not interesting for us as they cannot give rise to nonzero charge density: $\phi^\dagger Q \phi = 0$ simply because of the antisymmetry of the charge matrix Q.